A Model to Predict Spray-tip Penetration for Time-varying Injection Profiles

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Abstract

A new model to predict spray-tip penetration for a time-varying injection profile has been formulated based on gas-jet theory. The approach involves using an effective injection velocity for the spray tip based on a representative spray response time. It is assumed that the instantaneous injection velocity affects the spray tip with an exponential response function and that the response time is the particle residence time, consistent with the theory of translation of jet vortex rings from Helmholtz’s vortex motion analysis [11]. This Helmholtz theory is also shown to yield the well-known velocity decay rate of turbulent gas jets. A Duhamel superposition integral is used to determine the effective injection velocity for time-varying injection rates. The model is tested with different injection profiles and different ambient densities. The results are also compared with numerical results from a CFD code that has been calibrated for spray simulations. The comparisons agree very well and the new spray penetration model offers an efficient method to predict penetrations. The model also can be used to predict equivalence ratio distributions for combusting sprays and jets.

Keywords: Sprays, Droplet, Super-position integral, Vortex Motion, CFD

Introduction:

Spray models are extensively used in the modeling and simulation of various combustion systems, such as Internal Combustion engines and Gas Turbine engines. Currently available zero-dimensional combustion modeling codes use spray penetration models which are based on a quasi-steady assumption [1, 2]. These spray models are a direct extension of gas-jet theory based on steady injections [3, 4, and 5]. Such gas-jet theory-based models have been used extensively to determine the spray-tip penetration for steady-state injections by various researchers [3, 4]. The results have been found to agree well with experiments and fine-mesh CFD computations. However, realistic injection profiles are usually time-varying and thus steady-state gas-jet theory cannot be applied directly. CFD modeling of sprays and jets provides good predictions of tip-penetration, but as the mesh size is increased to be compatible with practical engine computations, the prediction accuracy becomes poorer. Hence, there is a need for a better predictive model for spray penetration that can be used with practical time-varying injection velocity profiles and realistic engine ambient conditions. In this work, we present a new method, which is based on jet-theory and a superposition integral formulation to determine an effective injection velocity and hence, the spray-tip penetration.

There have been attempts to study non-stationary jets and sprays. Measurements by Borée et al. [6], involved a study of a sudden decrease in injection velocity, and they proposed a self-similar result based on a temporal scaling. They found that a time-scale of the form $x/U_{inj,2}$, where $x$ is the position of the spray tip and $U_{inj,2}$ is the suddenly decreased injection velocity, leads to a self-similar result. However, their results and model only apply to a sudden decrease in injection velocity from an initial constant value to a constant lower value. Previous works pertaining to simple one-dimensional spray models include the packet penetration model of Desantes et al. [7], where an injected spray particle instantly travels with a different momentum once it is overtaken by a speeding subsequently injected particle. This approach is an improvement over other quasi-steady state models in the literature, but their study did not include a wide range of tested injection profiles. Zhang et al. [8] studied the effects of flow acceleration on turbulent jets with linear, quadratic and exponential injection profiles using measurements. They found that the temporal evolution of the spray front follows the same form as the forcing function at the nozzle. Wan et al. [9] also modeled spray penetration for evaporating sprays by using scaling parameters. They showed their model worked well for a linearly increasing injection velocity profile. However, they did not consider other injection velocity rate shapes. Breidenthal [10], in a study of self-similar, turbulent jets postulated that if the flow is non-stationary or non-steady a choice of self-similarity function is the exponential function. This self-similarity function can be thought of as the response function of a particle in the jet to a change in injection velocity. Crowe et al. [11] studied the response function of a droplet to the surrounding gas velocity. Their analysis also reveals an exponential response function of time for the droplet to reach the surrounding gas velocity.

In this work we propose a relatively simple explicit model that can be used to predict spray tip penetration.
The model formulates the effective injection velocity experienced at the spray tip at any given instant to determine the spray tip penetration. The model extends the isolated drop theory proposed by Crowe et al. [11] that a droplet requires time to adjust to a change in surrounding gas velocity. This change is based on a response function which is exponential in time and the response time is the ratio of a characteristic length scale to the surrounding gas velocity. The same argument is extended here to sprays where the characteristic length scale is taken as the location of the particle from the injector tip and velocity is the time-convolved injection velocity. This is derived from the classical work of Helmholtz's vortex motion theory [12-15], wherein in the translational velocity of a vortex ring is related to the circulation of the vortex. The present proposed model is tested with different time-varying injection profiles and varying ambient densities. The results compare very well with CFD results. It is to be noted that the choice of selecting CFD simulations as the benchmark for comparison with the proposed analytical model was based on the ability to test the model over wide ranges of injection profiles, which experimentally is a difficult and time-consuming task. The selected injection profiles include smooth profiles, profiles featuring a sudden decrease and sudden increase in velocity, and their combinations.

This paper is organized as follows. A detailed analysis including the derivation of the new model is provided first. A short description of the CFD and spray model [16] and details of the computational domain follows. In the same section we also discuss modifications to previous penetration correlations [1] as applied to non-stationary and non-steady jets and sprays. Analytical expressions for an injection profile with rising and falling rates are then derived based on an assumed average response-time. The analytical expression is also extended to consider the case of a sinusoidal injection profile. Next, we present 13 different injection cases to validate the proposed model. Finally, discussion about the use of the model for analyzing vaporizing sprays and the usefulness of the model is presented.

Theory:

The steady-state solution for the velocity profile within a turbulent gas jet can be obtained using similarity analysis of the 2D Navier-Stokes equations as [3, 4]:

$$U( x, r ) = \min \left( U_{\text{inj}} , \frac{3U_{\text{no}}^2d_{\text{eq}}}{2\pi r^2} \right) \left( 1 + \frac{3U_{\text{inj}}^2d_{\text{eq}}^2}{256\pi^2 r^2 x^2} \right)$$  \hspace{1cm} (1)

where $U_{\text{inj}}$ is the injection velocity at the injector exit. $x$ is the axial distance of a particle from the injector tip and $r$ is the radial distance of the parcel from the spray symmetry axis. $d_{\text{eq}}$ is the equivalent or effective diameter of the gas jet defined as [3]:

$$d_{\text{eq}} = \frac{d_{\text{no}}}{\sqrt{\rho_t / \rho_g}}$$  \hspace{1cm} (2)

where $d_{\text{no}}$ is the nozzle diameter and $\rho_t$ and $\rho_g$ are the injected and surrounding fluid densities, or in present study, the liquid and gas-phase densities, respectively. In the case of a turbulent jet $\nu_t$ is the turbulent viscosity given by [3]:

$$\nu_t = C_t \pi^{0.5} U_{\text{inj}} d_{\text{eq}} / 2$$  \hspace{1cm} (3)

$C_t$ is a constant, as reported by Abraham [3] and Schlichting[17] who used $C_t = 0.0161$. For evaluating spray tip penetration, the centerline tip position can be evaluated by setting $r=0$ in Eq. (1) and generalizing the velocity decay rate at the centerline as:

$$U( x ) = \frac{dx}{dt} = \frac{3U_{\text{no}} d_{\text{eq}}}{K x} \left( x \geq 3d_{\text{eq}} / K = x_0 \right)$$  \hspace{1cm} (4)

where $K$ is the entrainment constant and is equal to $16\pi^{1.5}C_t$. Schlichting’s choice of $C_t=0.0161$ gives $K=0.457$. If we consider $x$ to be the spray tip position at all times, then integrating Eq. (4) provides spray tip penetration as a function of time. For a constant injection velocity the analytical solution of spray tip position as a function of time is given as:

$$x(t) = \left( \frac{6}{K} \right)^{1/2} \left( U_{\text{no}} d_{\text{eq}} \right)^{1/2} t^{1/2} \text{ for } x \geq x_0$$  \hspace{1cm} (5)

where the entrainment constant, $K$ is assumed to be 0.5 in this study to give favorable agreement with experimental data. For cases with time-varying injection velocity, an analytical expression for the spray tip penetration is derived below.

A model of an isolated droplet responding to a given surrounding gas velocity was given by Crowe et al. [11]. The equation of motion for a particle in a gas can be expressed by the balance of drag force to the change of momentum of the particle as:
in Fig. 1, the counter rotating vortices are due to the jet tip. For a given spray angle, the eddy size at the tip position is assumed to be related to the vortex ring at the leading tip of the jet. If \( \Gamma \) is the total circulation around the core, a mean jet radius is \( L_{\text{eddy}} \), and the cross-sectional radius of the vortex ring is \( r_v \), then the velocity of the translation of the vortex ring, \( U_y \) is expressed as [12]:

\[
U_y = -\frac{\Gamma}{4\pi L_{\text{eddy}}} \left[ \ln \left( \frac{8L_{\text{eddy}}}{r_v} \right) - \frac{1}{4} \right] - \frac{\Gamma}{L_{\text{eddy}}} \tag{11}
\]

According to Helmholtz's second vortex theorem [13], circulation \( \Gamma \) is the same for all cross sections of the vortex tube (here vortex ring) and is independent of time. If we assume that the vortex ring expands only in its cross-sectional area, then the circulation \( \Gamma \) around the core of the vortex at time \( t \) is equal to the circulation when the ring was generated at the injector tip at time \( t=0 \), and is given [13, 14] as:

\[
\Gamma (x) = \frac{4U_e(x)}{\pi} L_{\text{eddy}} = \Gamma_{\text{inj}} = \frac{4U_{\text{inj}} d_{\text{neq}}}{\pi} - U_{\text{inj}} d_{\text{neq}} \tag{12}
\]

where the circulation \( \Gamma(x) \) is the circulation of the ring at location \( x \) and \( \Gamma_{\text{inj}} \) is the circulation of the vortex at the injector location, and it is written as a function of the injection velocity \( U_{\text{inj}} \) and the nozzle diameter \( d_{\text{neq}} \). If we assume that the translational velocity of the vortex is given by a ratio of a characteristic length scale \( \sim (L) \) and time-scale \( \sim (T) \), then from Eqs. (11) and (12) we have:

\[
U_y \sim \frac{L}{T} \sim \frac{\Gamma}{L_{\text{eddy}}} - \frac{U_{\text{inj}} d_{\text{neq}}}{x \tan(\theta)} \tag{13}
\]

Reitz and Bracco [18] predicted the variation of spray-half angle \( \theta \) and concluded that \( \tan(\theta) \sim (\rho \phi \mu) \sqrt{L/D} \). Using the correlation of \( \tan(\theta) \) and the effective diameter \( d_{\text{eq}} \), Eq. (13) can thus be re-written as:

\[
U_y \sim \frac{L}{T} \sim \frac{U_{\text{inj}} d_{\text{eq}}}{x \tan(\theta)} = \frac{d_{\text{eq}}}{x/U_{\text{inj}}} \tag{14}
\]

Note that this result, derived independently from Helmholtz’ vortex theory, is identical in form to the equation for the gas jet velocity decay of Eq.(4), when the vortex translational velocity equals the jet tip velocity. The only time scale that appears in the above equation is \( (x/U_{\text{inj}}) \) and hence it is reasonable to assume that the response time of the eddy to a particular injection velocity at the nozzle exit is of the order of the convective time-scale or flow residence time scale, \( \tau_F = (x/U_{\text{inj}}) \). This is also valid in the case of an injection velocity that is changing with time. Crowe et al. [11] argued in the case of droplets, that the response time is also proportional to the flow-time scale and the two time scales were related by the Stokes number.
Introducing the Stokes number, $St = \tau_e/\tau_r$, where $\tau_r$ is a characteristic time of the flow, which in the spray case is $x/U_{inj}$, following Crowe’s model [11], the response time of the particle, $\tau_e$ is given as:

$$\tau_e = St \frac{x}{U_{inj}} \quad (15)$$

This form of the time response is also consistent with that introduced in the experimental study of Borée et al. [6]. In the present study the Stokes number is taken as an experimental constant assumed to be $St=3$ and the effective injection profile of the spray tip corresponding to an injection velocity change from $U_1$ to $U_2$ is thus:

$$U_{inj,eff} = U_1 + \left( U_2 - U_1 \right) \left[ 1 - \exp \left( \frac{t - t_{birth}}{\tau_e} \right) \right] \quad (16)$$

where $t_{birth}$ is the time at which the injection velocity changes from $U_1$ to $U_2$. This effective injection velocity is then used in Eq. (4) to determine the spray tip penetration. For a time-varying injection where $U_{inj}$ changes $n$ times, the Duhamel superposition principle [19] provides the effective injection velocity for the penetration as a function of time as:

$$U_{inj,eff}(x,t) = U(0) + \sum_{k=1}^{n} A(x,t - t_k) \left( \frac{\Delta U}{\Delta t} \right) (\Delta t)_k$$

$$\Rightarrow U_{inj,eff}(x,t) = U(0) + \sum_{k=1}^{n} A(x,t - t_k) (\Delta U)$$

(17)

where $x$ is the spray tip position, $U(0)$ is the initial injection velocity at time $t=0$, $t_k$ is the birth time of the new injection velocity $U(k)$, and $(\Delta U)_k = U_{inj} - U_{inj-1}$, i.e., the change in injection velocity at $t=t_k$. The function $A(x,t-t_k)$ accounts for the exponential response and is given as:

$$A(x,t-t_k)=1-\exp \left( \frac{t-t_k}{\tau_{ek}} \right)$$

where $\tau_{ek}$ is the response time associated with the eddy at the spray tip at time $t=t_k$ and is given as:

$$\tau_{ek} = St \frac{x}{U_{k}} \quad (19)$$

Note that the summation in Eq. (17) can be replaced by integrals in the case of a continuous function $U_{inj}(t)$. The spray tip position is determined using the effective injection velocity in Eq. (4) as:

$$\frac{dx}{dt} = \frac{3}{K} \frac{U_{inj,eff}(x,t) \ d_{eq}}{x} \quad (x \geq x_0) \quad (20)$$

Ricou and Spalding [20] found in their experiments that the entrainment constant, $K$, increases as the injection Reynolds number is reduced. However, for simplicity, we assume a constant value of $K=0.5$. The above model is tested below with different injection profiles. At each step during the integration, the effective injection velocity at the spray tip position is evaluated from Eq. (17) and the new spray tip position is evaluated by integrating Eq. (20) with fourth-order Runge-Kutta integration.

**Modification to Spray-tip Penetration Correlations:**

Practical spray-tip penetration models in zero-dimensional or Multi-Zone Combustion models [1, 2] use correlations such as that of Hiroyasu and Arai [21]:

Before breakup,$0 < t < t_b$:

$$S = 0.39 \left[ \frac{2\Delta P}{\rho_f} \right]^{0.5} \quad t$$

After breakup,$0 < t < t_b$:

$$S = 2.95 \frac{\Delta P}{\rho_f} \left( \frac{d_{eq}}{t} \right)^{0.5}$$

(21)

where the breakup time, $t_b$, is:

$$t_b = 28.65 \frac{\rho_f d_{eq}}{\rho_g \Delta P^{0.5}}$$

(22)

Eq. (22) is similar to Eq. (5), except that the constant 2.48 is different, viz., \( \sqrt{\frac{6}{K}} = \sqrt{\frac{6}{0.5}} = 3.46 \). Thus, the Hiroyasu penetration correlation predicts quantitatively a slightly lower spray tip penetration than the present model. The breakup time can also be compared to the ratio of $x_0/U_{inj}$ and is given as:

$$t_b = \frac{x_0}{U_{inj}} = \frac{3}{K} \frac{d_{eq}}{U_{inj}} = \frac{3}{2K} \left( \frac{\rho_f d_{eq}}{\rho_g \Delta P^{0.5}} \right)$$

(23)

This constant in the modified correlation is 3 as compared to the constant of 28.65 in the Hiroyasu’s correlation. This indicates that the apparent origin of the jet occurs further downstream of the nozzle in the case of sprays, reflecting the fact that the jet breakup process takes time to occur.
Approximate Analytical Solutions for Simple Injection Profiles:

The model presented above evaluates spray-tip penetration based on Eqs. (17) to (20). Approximate analytical expressions can be obtained by assuming quasi-steady average response times to evaluate the velocity field. Consider an injection velocity profile of the form \( U_{inj}(t) = a + mt \), where \( a \) is the initial injection velocity and \( m \) is the slope of injection velocity rate shape. For simplicity it is assumed that the response time, \( \tau_{v,av} \) is constant. Simplifying Eq. (17) in integral form gives:

\[
U_{inj,eff}(x,t) = U(0) + \int_{0}^{x/a} \left[ 1 - \exp \left( \frac{t - t'}{\tau_{v,av}} \right) \right] \left( \frac{dU(\tau)}{d\tau} \right) d\tau
\]

\[
\Rightarrow U_{inj,eff}(x,t) = U(t) - \int_{0}^{x/a} \exp \left( \frac{t - t'}{\tau_{v,av}} \right) (U') \, dt
\]

(24)

where \( U' \) denotes the time differential and \( \tau_{v,av} \) is an averaged response time modeled as:

\[
\tau_{v,av}(t) = \frac{St_{d}}{U_{AV}} \quad U_{AV} = \int_{0}^{T} \rho_{m} A_{m} U_{m}^{2} \, dt
\]

(25)

where \( T \) is the injection duration and \( U_{AV} \) is the mass-averaged injection velocity. \( St_{d} \) is viewed as an adjustable constant chosen here as 50. This high value is consistent with the constant used in the Hiroyasu correlation Eq. (23). From Eq. (15), the length scale \( x \) is the distance from the particle location to the injector nozzle, which is dynamically calculated over the time. Since a-priori information about the particle position in the approximate analytical approach is unavailable, the length scale chosen is the effective nozzle diameter.

Using \( U' = n_{c} \), Eq. (24) gives an effective injection velocity for a linear injection profile as:

\[
U_{inj,eff}(x,t) = (a + mt) - m \, \tau_{v,av} \left( 1 - \exp \left( \frac{t}{\tau_{v,av}} \right) \right)
\]

(26)

Substituting Eq. (26) in Eq. (20) and solving for \( x \) gives the spray-tip penetration as:

\[
x(t) = \left[ \frac{3D_{cr}}{K} \left( \frac{2(a - m \tau_{v,av})}{a + mT} + mt^{2} \right) \right]^{1/2}
\]

(27)

where for a particular linear-injection profile, \( \tau_{v,av} \) is given from Eq. (25) as:

\[
\tau_{v,av}(a, m, T) = St_{d} \, d_{eq} \left[ \frac{3mT \left( a + \frac{1}{2} mT \right)}{(a + mT)^{2} - a^{2}} \right]
\]

(28)

Note that Eq. (27) reduces to Eq. (4) if \( m=0 \) and \( a=U_{inj} \) (top hat injection profile).

Similar approximate solutions can also be found for other profiles such as for a sinusoidal injection profile of the form \( U_{inj}(t) = C \sin(\alpha t) \), where \( \alpha = 2\pi/T \) and \( T \) is the total duration of injection. Using \( U(t) = \alpha \cos(\alpha t) \) and Eq. (24), the effective injection velocity is predicted to be:

\[
\frac{C\alpha \tau_{v,av}}{1 + \alpha^2 \tau_{v,av}^2} \left( \cos(\alpha t) + \alpha \tau_{v,av} \sin(\alpha t) - \exp \left( -\frac{t}{\tau_{v,av}} \right) \right)
\]

(29)

where for a particular sinusoidal-injection profile, \( \tau_{v,av} \) is given by Eq. (25) as:

\[
\tau_{v,av}(C, \alpha, T) = St_{d} \, d_{eq} \left[ \frac{2(\cos(\alpha T) - 1)}{C \alpha T - \frac{1}{2} \sin(2\alpha T)} \right]
\]

(30)

The spray-tip penetration for a sinusoidal injection profile is then obtained by substituting the effective injection velocity obtained above in Eq. (20) and solving for \( x \), yielding:

\[
x(t) = \left[ \frac{l}{\alpha + \alpha^2 \tau_{v,av}^2} \right]^{1/2} \left[ \frac{6CD_{m}}{\tau_{v,av}^{2} \alpha \sin(\alpha t) - \left( \frac{1}{l} + \frac{\alpha^2 \tau_{v,av}^2}{K} \right) \exp \left( -\frac{t}{\tau_{v,av}} \right) \right] \right]^{1/2}
\]

(31)

Fuel-air distribution in Evaporating Sprays:

An additional application of the present model is to consider spray combustion. For steady-injections under evaporating conditions, the equivalence ratio at each axial location in the spray is defined as the ratio of the air mass flux to the fuel mass flux at that location [5] to the stoichiometric air/fuel ratio. For unsteady injections the effective injection velocity can be used to determine the equivalence ratio distribution. For example, assuming a stoichiometric air/fuel ratio of 15, the local equivalence ratio at position \( x \) is:
Figure 3 shows the injection profiles with a descriptive sudden increases or decreases in injection velocity. Combinations of smooth functions and profiles with the model was tested at two different ambient densities.

A total of 13 different injection profiles were chosen. Details of Test Cases:

The spray-tip penetration results from the present zero-dimensional model in Eq. (20) are compared with the results from the corresponding CFD simulations. The comparisons are presented in Figs. 4 to 29. Each figure shows a comparison of spray tip penetration predicted from the zero-dimensional model (solid line) and the CFD simulation (circles). The plots also show the actual injection velocity profile (dashed line) and the effective injection velocity profile as experienced by the spray tip (dotted line). The plots also show the error in the penetration as a function of time. The error is calculated from:

\[
\text{Error} = 100 \frac{\sqrt{\left(S_{od} - S_{cfd}\right)^2}}{S_{cfd}}
\]

where \(S_{od}\) and \(S_{cfd}\) are the spray tip penetration from the 0-D model and CFD simulations, respectively. The relative error can be high initially when the absolute penetration is low. However as the spray progresses, the error reduces with time. Thus, the error is evaluated starting only after 0.2 ms from the beginning of injection to avoid the anomalous near nozzle region. Figures 4, 5 and 6 show the flat injection velocity profiles for the chamber density of 60.6 kg/m³. It can be noted that the error for intermediate and high injection velocities is larger than for the low injection velocity, although the mean error is less 5%. Also, the effective injection profile at the tip coincides with the actual injection profile (see Eq. (17)). Figure 7 corresponds to the case with a peak in the injection profile. Note that the zero-dimensional model follows the trend of the spray tip penetration during the change in injection velocity. The injection velocity does not increase suddenly, as would occur with a quasi-steady state model, but instead gradually increases following the rise in injection velocity. A similar trend is observed during the decreasing portion of the velocity profile. Figures 8 shows the sinusoidal injection profile and the penetration appears to be very close to the numerical result with reduced error as time progresses. Figures 9, 10 and 11 show the results from the STEP, STEP3 and STEPR3 injection profiles. The penetration is seen again to compare well with the numerical results, but
with slight differences towards the end. It can be noted that the slopes of the penetration curves are similar to those obtained from the CFD results.

Figure 12 shows case PEAK2, where the peak exists for a shorter duration but with a higher injection velocity during the rise as compared to case PEAK. As can be seen, a more severe high peak also agrees well with the CFD predicted penetration. Figures 13, 14 and 15 represent pulsed injection cases (PULSE1, PULSE2 and PULSE3), respectively. In the case of multiple pulses of equal pulse duration and equal pulse height (viz., PULSE1 and PULSE3), the effective injection velocity oscillates about a particular velocity. Figure 14 is a one pulse case. The tip penetration from the new model compares well with the spray-tip penetration from the CFD simulation and demonstrates an exponentially reducing the effective injection velocity at the spray-tip. Finally, the triangular injection profile case is shown in Figure 16. The penetration again compares well with the CFD prediction.

Similar results were obtained with a lower chamber density of 30.2 kg/m$^3$, and the comparison of the spray tip penetration with the zero-dimensional model and the CFD simulations agreed well for most of the cases. Only cases PEAK2 and SINE shown in Figs. 17 and 18 had significant errors in penetration. The SINE case shows that the penetration is under predicted from about 0.7 ms after the start of injection. This show that choice of the $St=3$, after 0.7 ms of injection gives a relatively slower response-time and hence, a relatively lower effective injection velocity in the first half which also effect the penetration in the second half of injection. In case of PEAK2, the injection velocity changes from about 280 m/s to 690 m/s. The penetration compares well with the CFD result during the duration when injection velocity is increasing. However during the period when the injection velocity is decreasing the penetration is under-predicted which is due to a relatively lower effective injection velocity. This show that the choice of $St=3$ results in a relatively faster time response during this period of injection. Hence, these two cases show that for some particular injection cases, a variable St depending upon the magnitude of the injection velocity change could help in improving result. The agreement could also have been improved by considering a variable entrainment constant, as suggested by Ricou and Spalding [13], or with a variable Stokes number for the droplet momentum response. Such tuning of the model could be considered as a future improvement for wider ranges of injection conditions, such as for the case of the sudden large increase in injection velocity of Fig. 12.

It is also of interest to explore results obtained with the analytical spray tip penetrations derived for the linear and sinusoidal injection profiles in Eqs. (27) and (31). Spray-tip penetration plots and the effective injection velocity profile for the line-shape injection cases are shown in Figs. 19 and 20 for positive and negative slopes $U_{ij} = a + mt$, where $a$ is the injection velocity at $t=0$ and $m$ is rate of change of injection velocity, respectively.

Figure 19 shows the case with positive slopes. Figure 19(a) shows the comparison of penetration predictions from the zero-dimensional model, CFD model and the analytical model. The penetration predictions from analytical model agree well with both the CFD model and the zero-dimensional model. Figure 19(b) compares the prediction of the effective injection velocity profile at the spray tip from the zero-dimensional model and analytical model with that of the actual injection velocity profile. The effective velocity profile from the analytical model is observed to be greater than that of the zero-dimensional model. However its effect on the penetration prediction is less, as observed from Fig. 19(a). Similar plots are shown for the case with negative slope of the linear injection profile in Figs. 20(a) and 20(b). From Fig. 20 (b) it can be observed that the prediction of the effective injection velocity at the tip from the analytical model matches well with that of the CFD and zero-dimensional models. The penetrations also match very well as shown in Fig. 20(a). Figures 21(a) and 21(b) show similar plots for the sinusoidal case and show a good match with the CFD results and the zero-dimensional model. Again it can be observed from Fig. 21(b) that the effective injection velocity at the spray tip from the analytical model is under-predicted compared to that of the zero-dimensional model, but the penetrations agree well, as shown in Fig. 21(a). Hence, it be concluded that for the linear and sinusoidal injection profiles, the approximate model to predict the effective injection velocity profile can provide good predictions of spray tip penetration.

Finally, Fig. 22 shows a plot of the variation of equivalence ratio at the spray tip with time for four different cases, viz., the linear injection profile with positive and negative slopes, and the sinusoidal cases as discussed above analytically, plus a constant injection velocity case having an average velocity approximately equivalent to the former three cases (385 m/s). The equivalence ratio is analytically calculated from Eq. (32). Figure 22 shows the comparison of the four cases. It can be observed that the constant injection case and the linearly decreasing case exhibit similar trends of variation of $\phi$. The plot also shows a change in slope in the near nozzle area, which is due to the change in slope of the penetration, $x$ in the near nozzle area. This is
consistent with the observations of Naber and Siebers [5] and with Eqs. (21) where in the near nozzle area, penetration is proportional to $t$ and in the far-field proportional to $t^{1/2}$. This trend is not visible for the other two cases of the linearly increasing profile and the sinusoidal profile. This is because these profiles start with a very low injection velocity, which is not significant enough to reveal the changes in the gradient in the near-field. The plot also shows the time when the equivalence ratio at the spray tip reaches unity. For the linear profile with negative and positive slopes, the time to reach $\phi=1.0$ is $\tau_{m} = 0.05$ ms and $\tau_{m+} = 0.26$ ms, respectively. Similarly for constant injection and sinusoidal injection profiles, the time to reach $\phi=1$ is $\tau_{const} = 0.07$ ms and $\tau_{sine} = 0.37$ ms, respectively. Thus, $\tau_{m} \sim \tau_{const} < \tau_{m+} < \tau_{sine}$. Hence, it can be deduced that for a similar mass average injection velocity, the spray tip reaches a combustible equivalence ratio early for constant injection cases and for linearly decreasing cases, and later for linearly increasing cases and even later for a sinusoidal rate shape. This suggests that linearly increasing or the sinusoidal injection profiles could potentially be a choice for late combustion regimes and linearly decreasing and constant injection velocity profiles are best suited for early combustion regimes for diesel engines.

**CONCLUSION:**

A new zero-dimensional model has been formulated to evaluate spray-tip penetration for time-varying injection velocity profiles based on gas-jet theory that is consistent with Helmholtz’s vortex-model. The new model formulates an effective injection velocity for particles located at the spray tip and exhibits an exponential response function to changes in the injection velocity. A Duhamel superposition integral is used to determine the effective injection velocity at the spray tip for time-varying cases. The model is tested with a wide variety of injection profiles and different ambient densities. The results compare very well with numerical results from CFD simulations. The new spray model can be used to improve current zero-dimensional engine codes and other applications where accurate spray tip penetration prediction is required.

Analytical results for linear and sinusoidal injection velocity profiles are also derived. The analytical results compare well with the CFD results and the zero-dimensional model. The model can also be applied to predict the fuel-air distribution within evaporating sprays. The equivalence ratio distributions for four different injection profiles are compared with the analytical model. The results show significant influence of the choice of the injection profiles on the time for the spray tip region to reach an equivalence ratio of unity.

For late combustion regimes, linearly increasing and sinusoidal profiles are helpful to delay the mixing while for early combustion, constant and linearly decreasing injection profiles are best suited. The new spray model thus offers a predictive tool for the selection of injection profiles based on the mixing criteria in this example application.

**Acknowledgements:**

Authors would like to acknowledge funding support from the DOE Sandia Labs.

**References:**


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<th>Mean</th>
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Table 1: Error in predicted spray tip penetration for $\rho_g = 60.6 \text{ kg/m}^3$.

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<th>% Error ($\rho_g = 30.2 \text{ kg/m}^3$)</th>
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Table 2: Error in predicted spray tip penetration for $\rho_g = 30.2 \text{ kg/m}^3$.  

Figure 1: Schematic describing the momentum response time for the spray-tip eddy to adjust to the injection velocity.

Jet showing counter rotating vortices at the spray tip and Vortex Ring

\[ \tau_v = \frac{X}{U_{inj}} \]

\[ L_{eddy} = X \tan(\theta) \]

Radius of cross section = \( r_a \)

Injector Location

100 mm; \( N_x = 400 \)

40 mm; \( N_r = 160 \)

Figure 2: Details of computational domain and mesh size for CFD simulations.
Figure 3: Nomenclature of the 13 injection profiles tested.
Figure 4: Spray Tip Penetration for LOWFLAT case, $\rho_g \approx 60.6 \text{ kg/m}^3$.

Figure 5: Spray Tip Penetration for FLAT case, $\rho_g \approx 60.6 \text{ kg/m}^3$.

Figure 6: Spray Tip Penetration for HIGHFLAT case, $\rho_g \approx 60.6 \text{ kg/m}^3$.

Figure 7: Spray Tip Penetration for PEAK case, $\rho_g \approx 60.6 \text{ kg/m}^3$. 
Figure 8: Spray Tip Penetration for PEAK case, \( \rho_g \approx 60.6 \text{ kg/m}^3 \).

Figure 9: Spray Tip Penetration for STEP case, \( \rho_g \approx 60.6 \text{ kg/m}^3 \).

Figure 10: Spray Tip Penetration for STEP3 case, \( \rho_g \approx 60.6 \text{ kg/m}^3 \).

Figure 11: Spray Tip Penetration for STEPR3 case, \( \rho_g \approx 60.6 \text{ kg/m}^3 \).
Figure 12: Spray Tip Penetration for PEAK2 case, $\rho_g \approx 60.6 \text{ kg/m}^3$.

Figure 13: Spray Tip Penetration for PULSE1 case, $\rho_g \approx 60.6 \text{ kg/m}^3$.

Figure 14: Spray Tip Penetration for PULSE2 case, $\rho_g \approx 60.6 \text{ kg/m}^3$.

Figure 15: Spray Tip Penetration for PULSE3 case, $\rho_g \approx 60.6 \text{ kg/m}^3$. 
Figure 16: Spray Tip Penetration for TRIANGULAR case, $\rho_g \approx 60.6 \text{ kg/m}^3$.

Figure 17: Spray Tip Penetration for SINE case, $\rho_g \approx 30.2 \text{ kg/m}^3$.

Figure 18: Spray Tip Penetration for PEAK2 case, $\rho_g \approx 30.2 \text{ kg/m}^3$.

Figure 19: Spray Tip Penetration for line-shaped case (positive slope), $\rho_g \approx 60.6 \text{ kg/m}^3$. 

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Figure 20: Spray Tip Penetration for line-shaped case (negative slope), \( \rho_g \approx 60.6 \text{ kg/m}^3 \).

Figure 21: Spray Tip Penetration for sinusoidal-shaped case, \( \rho_g \approx 60.6 \text{ kg/m}^3 \).

Figure 22: Equivalence Ratio at Spray-tip position for linear and sinusoidal-shaped injection rate cases, \( \rho_g \approx 60.6 \text{ kg/m}^3 \).